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Correlation Functions of the Scalar Field in Background NC U(1) Yang-Mills.

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Abstract

We consider the complex scalar field coupled to background NC U(1) YM and calculate the correlator of two gauge invariant composite operators. We show how the noncommutative gauge invariance is restored for higher correlators (though the Green's function itself is not invariant). It is interesting that the recently discovered noncommutative solitons appear in the calculation.

1 Introduction

Noncommutative gauge theories have attracted attention due to recent achievements in string theory. Seiberg and Witten in their celebrated work have shown that the open string theory in large external constant B-field yields the effective noncommutative theory on the world volume of D-brane[2]. That is why coupling of a noncommutative theory to a commutative one can be useful for understanding the interaction of open and closed strings.

It is the fundamental idea that the noncommutativity is a deformation of the algebra of functions, thus the fields of a noncommutative gauge theory are valued in the deformed algebra of functions where the usual product is replaced by the associative noncommutative Moyal $*$-product

$$f_1 * f_2(x) = f_1(x)\mathrm{e}^{\frac{\mathrm{i}}{2}\overleftarrow{\partial}_\mu \theta^{\mu\nu}\overrightarrow{\partial}_\nu} f_2(x). \tag{1}$$

The following integral representation of the $*$-product will be used in the future,

$$\begin{aligned} f_1 * f_2(x) &= \int d^dx'\, d^dx''\, K(x,x',x'')f_1(x')f_2(x''), \\ K(x,x',x'') &= \frac{1}{\pi^d \det(\theta^{\mu\nu})}\exp\{-2\mathrm{i}(x'-x)^\mu(\theta^{-1})_{\mu\nu}(x''-x)^\nu\}. \end{aligned} \tag{2}$$

We shall restrict ourselves to the case of two noncommuting spatial coordinates,

$$\left[x^1, x^2\right] = \mathrm{i}\theta^{12} \equiv \mathrm{i}\theta.$$

In this case[5]

$$K(x,y,z) = \frac{1}{\pi^2\theta^2}\mathrm{e}^{-\frac{2\mathrm{i}}{\theta}\left(x^2(y^1-z^1)+y^2(z^1-x^1)+z^2(x^1-y^1)\right)}. \tag{3}$$

This formula will be exploited in the subsequent calculations to evaluate different $*$-products. If some extra commuting temporal or spatial coordinates are present they serve as additional parameters from the $*$-product view-point.

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The Weyl-Moyal (WM) correspondence is a very useful tool for performing calculations in noncommutative theories. This is the one-to-one correspondence (isomorphism) between the deformed algebra of functions (i.e. the former algebra to which noncommutative fields belong) on a noncommutative manifold with a constant noncommutativity matrix θ^{ij} and the algebra of operators acting in the auxiliary Hilbert space. The literature on this topic is very extended so we do not spend too much time for discussing it (for example, see[1]).

The paper is organized as follows. In the next section we present the model, i.e. its action functional and gauge transformations. This is the massive charged scalar field ϕ coupled to a background noncommutative gauge field A. Further we consider a particular background of constant strength. The third section deals with the correlation functions of the theory. Such a free (quadratic in ϕ) theory appears to be in some sense equivalent to an effective commutative one. This equivalence becomes especially transparent physically when we use the linear ansatz for the gauge field[4]. Namely, a gauge-dependent rescaling of coordinates is required to perform this reduction. That is why the definition of the gauge invariant correlators proves to involve the $*$-product and differs from that in the commutative case. It is the $*$-product that is expected to rescue the situation. The gauge invariance of these correlators is verified by the explicit calculation using the formal spectral definition of the Green's function. These objects are shown to really possess the required feature. The gauge invariant two-point function is calculated explicitly as an infinite series where each term is obviously gauge invariant.

2 Classical theory

The model we consider is the complex scalar field theory coupled to background noncommutative U(1) Yang-Mills. The action functional is given by

$$S = -\int \bar{\phi}(-D_i D^i + m^2)\phi. \tag{4}$$

The metric tensor is considered to be of euclidean (++) signature. The covariant derivatives act according to

$$\begin{aligned} D_i\phi &= \partial_i\phi - \mathrm{i}A_i * \phi, \\ F_{ij} &= \mathrm{i}\,[D_i, D_j] = \partial_i A_j - \partial_j A_i - \mathrm{i}[A_i, A_j]. \end{aligned} \tag{5}$$

The gauge transformations

$$\begin{aligned} &\phi \to U * \phi, \quad \bar{\phi} \to \bar{\phi} * \bar{U}, \\ &A_i \to U * A_i * \bar{U} - \mathrm{i}\partial_i U * \bar{U}, \\ &F_{ij} \to U * F_{ij} * \bar{U} \end{aligned} \tag{6}$$

are generated by a star-unitary U such that

$$\bar{U} * U = 1 = U * \bar{U}. \tag{7}$$

In what follows we shall work with the background of constant strength $F_{12} = F$ and choose the potential

$$\begin{aligned} A_1 = -\alpha_1 x^2, &\quad A_2 = \alpha_2 x^1, \\ F = \alpha_1 + &\alpha_2 + \theta\alpha_1\alpha_2. \end{aligned} \tag{8}$$

Though F itself is not gauge invariant, in the present situation it is so. $F > 0$ will be assumed hereinafter without loss of generality. Without analyzing in detail all the gauges[4] we simply exhibit the one-parametric family of the functions U that generate some gauge

transformations leaving the potential like (8) within this class:

$$\begin{aligned} U_t &= \frac{1}{\cosh t} \mathrm{e}^{\frac{2\mathrm{i}}{\theta} x^1 x^2 \tanh t}, \\ U_0 &= 1, \quad \bar{U}_t = U_{-t}, \\ U_{t_1} * U_{t_2} &= U_{t_1+t_2}. \end{aligned} \tag{9}$$

The gauge transformation generated by U_t changes α_i as

$$\begin{aligned} \alpha_1 &\to \mathrm{e}^{-2t}\alpha_1 - \frac{2}{\theta}\mathrm{e}^{-t}\sinh t, \\ \alpha_2 &\to \mathrm{e}^{2t}\alpha_2 + \frac{2}{\theta}\mathrm{e}^{t}\sinh t. \end{aligned} \tag{10}$$

Another interesting thing is that the invariance of the action (4) is provided even if [1]

$$\bar{U} * U = 1, \quad U * \bar{U} = 1 - P. \tag{11}$$

Obviously $P * P = P$, i.e. P is a projector. This generating function is 'topologically nontrivial', i.e. $U \neq \mathrm{e}_*^{\mathrm{i}f}$ for any real f. Field strength transforms according to

$$F_{ij} \to U * F_{ij} * \bar{U} + U * (A_j * \partial_i \bar{U} - A_i * \partial_j \bar{U}) * P + \mathrm{i}(\partial_i U * \partial_j \bar{U} - \partial_j U * \partial_i \bar{U}) * P \tag{12}$$

but this makes no trouble as the $F_{ij}F^{ij}$ term is not present in the action for background A_i. In the case of such a transformation the gauge field A_i no longer remains real.

3 Quantum theory

It has been emphasized that free noncommutative theories are identical to the commutative ones, so the definition of the free propagator in the background gauge field is the same, i.e. as the inverse to the quadratic in ϕ part of the action. If f_n are orthonormalized eigenfunctions of the operator $(-D_i D^i + m^2)$ with the corresponding eigenvalues λ_n then the Green's function is given by its formal spectral definition [2]

$$G(x_{(1)}, x_{(2)}) = -\sum_n \frac{1}{\lambda_n} f_n(x_{(1)}) \bar{f}_n(x_{(2)}), \tag{13}$$

n being discrete or continuous. As in the usual case,

$$\left\langle \phi(x_{(1)}) \bar{\phi}(x_{(2)}) \right\rangle = \int \mathcal{D}\phi\, \mathcal{D}\bar{\phi}\, \mathrm{e}^{\mathrm{i}S[\phi,\bar{\phi}]} \phi(x_{(1)}) \bar{\phi}(x_{(2)}) = \mathrm{i}G(x_{(1)}, x_{(2)}). \tag{14}$$

This free propagator is not gauge invariant and transforms as

$$G(x_{(1)}, x_{(2)}) \to U(x_{(1)}) * G(x_{(1)}, x_{(2)}) * \bar{U}(x_{(2)}). \tag{15}$$

To verify this we have to check the invariance of the measure w.r.t. the gauge transformations. Now it becomes clear why both conditions of (7) are necessary. An example of U satisfying (11) and not satisfying (7) is[6]

$$\hat{U} : |n\rangle \to |n+1\rangle. \tag{16}$$

(we have used the WM correspondence). If one now approximates the path integral measure as

$$\begin{aligned} \mathcal{D}\phi\, \mathcal{D}\bar{\phi} &= N \prod_{m,n \geq 0} d\phi_{mn} d\bar{\phi}_{mn}, \\ \phi_{mn} &= \langle m|\hat{\phi}|n\rangle, \quad \bar{\phi}_{mn} = \langle m|\hat{\phi}^\dagger|n\rangle \end{aligned} \tag{17}$$

[1] Author thanks A.Morozov for pointing at this fact.

[2] Upper indices of x's are coordinate indices and those in brackets will onwards mark the number of point in correlator.

where $\hat{\phi}$ denotes the operator to which ϕ is mapped under the Weyl-Moyal correspondence then

$$\langle m|\hat{U}\hat{\phi}|n\rangle = \begin{cases} \phi_{m-1,n}, & m \geq 1 \\ 0, & m = 0 \end{cases} \tag{18}$$

The similar formula holds for $\bar{\phi} \leftrightarrow \hat{\phi}^\dagger$ and even the domain of integration is not invariant. Another way to obtain the transformation law (15) is expansion (13). If the transformation generated by U is invertible, i.e. U has a left inverse that can generate a gauge transformation (in the case of (11) $\bar{U}$ cannot do it) then all the eigenfunctions before and after the transformation are in one-to-one correspondence and (15) becomes obvious.

If one chooses $A_1 = -Fx^2$, $A_2 = 0$, then $D_1 = (1 + \frac{F\theta}{2})\partial_1 + \mathrm{i}Fx^2$, $D_2 = \partial_2$. Thereby is the effect of noncommutativity merely rescaling the x^1 coordinate? On the other side, the gauge $A_1 = 0$, $A_2 = Fx^1$ implies rescaling of the other coordinate. Should one use the symmetric gauge, both coordinates would be rescaled by an identical factor (this situation seems to respect the rotational symmetry more than the two former ones). It becomes clear from this simple example that the subject of interest (in our case gauge invariant correlators) is also changed w.r.t. the commutative case. So the naive correlator $\langle :\bar{\phi}\phi(x_{(1)})::\bar{\phi}\phi(x_{(2)}):\rangle$ no more remains gauge invariant and should be replaced by [3]

$$\big\langle :\bar{\phi} * \phi(x_{(1)})::\bar{\phi} * \phi(x_{(2)}):\big\rangle. \tag{19}$$

Let us denote $\beta_i = 1 + \frac{\alpha_i\theta}{2}$, then the covariant derivatives take the form

$$\begin{aligned} D_1 &= \beta_1\partial_1 + \mathrm{i}\alpha_1 x^2, \\ D_2 &= \beta_2\partial_2 - \mathrm{i}\alpha_2 x^1. \end{aligned} \tag{20}$$

The problem of finding the above eigenfunctions f_n can be solved using the ansatz $f_n(x) = \exp(\mathrm{i}\frac{\alpha_2}{\beta_2}x^1x^2)g_n(x)$; it reduces to

$$\left\{-\beta_1^2\partial_1^2 - \beta_2^2\partial_2^2 - 2\mathrm{i}\big(\frac{\beta_1^2\alpha_2}{\beta_2} + \alpha_1\beta_1\big)x^2\partial_1 + \big(\frac{\beta_1^2\alpha_2}{\beta_2} + \alpha_1\beta_1\big)^2(x^2)^2 + m^2\right\} g_n = \lambda_n g_n. \tag{21}$$

From this one easily finds requisite eigenfunctions/eigenvalues

$$\begin{aligned} f_{n,k} &= \frac{\sqrt[4]{F}}{\sqrt{2\pi|\beta_2|}} \exp\left\{\mathrm{i}\big(\frac{\alpha_2}{\beta_2}x^1x^2 + kx^1\big)\right\}\psi_n\left(\frac{x^2\sqrt{F}}{\beta_2} + \frac{\beta_1 k}{\sqrt{F}}\right), \\ \lambda_n &= (2n+1)F + m^2. \end{aligned} \tag{22}$$

Here ψ_n stands for the n-th normalized wavefunction of one-dimensional harmonic oscillator with frequency equal to unity, i.e. $\mathrm{e}^{-\frac{x^2}{2}}$ multiplied by some Hermite polynomial. To obtain the correlator (19) one can use the ordinary Wick's theorem coming from the commutative case arriving to

$$\big\langle :\bar{\phi} * \phi(x_{(1)})::\bar{\phi} * \phi(x_{(2)}):\big\rangle = -G(x_{(1)}, x_{(2)})e^{\frac{\mathrm{i}}{2}\theta^{ij}(\overrightarrow{\partial}_{(1)i}\overleftarrow{\partial}_{(1)j} + \overleftarrow{\partial}_{(2)i}\overrightarrow{\partial}_{(2)j})}G(x_{(2)}, x_{(1)}). \tag{23}$$

The answer is

$$\begin{aligned} &\big\langle :\bar{\phi} * \phi(x_{(1)})::\bar{\phi} * \phi(x_{(2)}):\big\rangle = \\ &-\sum_{n_1,n_2=0}^{\infty}\frac{1}{\lambda_{n_1}\lambda_{n_2}}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} dk_1\,dk_2\big\{\bar{f}_{n_1,k_1} * f_{n_2,k_2}(x_{(1)})\big\}\big\{\bar{f}_{n_2,k_2} * f_{n_1,k_1}(x_{(2)})\big\}. \end{aligned} \tag{24}$$

[3] This correlator will be referred to as a two-point one.

The integral kernel (3) is of extreme use for the evaluation of the rhs of (24). The result is

$$\bar{f}_{n_1,k_1} * f_{n_2,k_2}(x) = -\frac{|2+\alpha_2\theta|\sqrt{F}}{4\pi|1+\alpha_2\theta|}\psi_{n_1}\left(\frac{(2+\alpha_2\theta)(2+F\theta)}{4(1+\alpha_2\theta)\sqrt{F}}k_1 + \frac{(2+\alpha_2\theta)F\theta}{4(1+\alpha_2\theta)\sqrt{F})}k_2 + x^2\sqrt{F}\right)$$
$$\psi_{n_2}\left(\frac{(2+\alpha_2\theta)F\theta}{4(1+\alpha_2\theta)\sqrt{F})}k_1 + \frac{(2+\alpha_2\theta)(2+F\theta)}{4(1+\alpha_2\theta)\sqrt{F}}k_2 + x^2\sqrt{F}\right) e^{ix^1\frac{(k_2-k_1)(2+\alpha_2\theta)}{2(1+\alpha_2\theta)}}. \tag{25}$$

In the future the following substitution will be useful:

$$\begin{aligned} k_1 &= \frac{\sqrt{F}}{2+F\theta+\alpha_1\theta}((2+F\theta)\xi_1 - F\theta\xi_2), \\ k_2 &= \frac{\sqrt{F}}{2+F\theta+\alpha_1\theta}(-F\theta\xi_1 + (2+F\theta)\xi_2), \\ \Big|\det\Big(\frac{\partial(k_1,k_2)}{\partial(\xi_1,\xi_2)}\Big)\Big| &= \frac{4F(1+\alpha_2\theta)}{(1+\alpha_1\theta)(2+\alpha_2\theta)^2}\,\mathrm{sign}(1+F\theta) \end{aligned} \tag{26}$$

as then

$$\bar{f}_{n_1,k_1} * f_{n_2,k_2}(x) = -\frac{|2+\alpha_2\theta|\sqrt{F}}{4\pi|1+\alpha_2\theta|}\psi_{n_1}\left(x^2\sqrt{F}+\xi_1\right)\psi_{n_2}\left(x^2\sqrt{F}+\xi_2\right)e^{ix^1\sqrt{F}(\xi_2-\xi_1)}. \tag{27}$$

As λ_n's do not depend on k, the integration over k_1, k_2 (or, equivalently, ξ_1, ξ_2) in (24) can be done explicitly and the result is proportional to (we have performed the constant shift in the integration variables $\xi_i \to \xi_i - \frac{(x^2_{(1)}+x^2_{(2)})\sqrt{F}}{2}$)

$$\begin{aligned} &\int d\xi_1\, d\xi_2\, \psi_{n_1}\left(\frac{x^2\sqrt{F}}{2}+\xi_1\right)\psi_{n_1}\left(-\frac{x^2\sqrt{F}}{2}+\xi_1\right) \\ &\times\psi_{n_2}\left(\frac{x^2\sqrt{F}}{2}+\xi_2\right)\psi_{n_2}\left(-\frac{x^2\sqrt{F}}{2}+\xi_2\right)\exp\left\{2i\frac{x^1\sqrt{F}}{2}(\xi_1-\xi_2)\right\}, \\ &x \equiv x_1 - x_2. \end{aligned} \tag{28}$$

The crucial feature is the definite parity of ϕ_n's as now

$$\int d\xi\, \psi_n\left(-\frac{x^2\sqrt{F}}{2}+\xi\right)\psi_n\left(\frac{x^2\sqrt{F}}{2}+\xi\right)\exp\left\{2i\frac{x^1\sqrt{F}}{2}\xi\right\} =$$
$$(-1)^n\int d\xi\, \psi_n\left(\frac{x^2\sqrt{F}}{2}-\xi\right)\psi_n\left(\frac{x^2\sqrt{F}}{2}+\xi\right)\exp\left\{2i\frac{x^1\sqrt{F}}{2}\xi\right\} = \frac{(-1)^n}{2}\phi_n\left(\frac{x^2\sqrt{F}}{2},\frac{x^1\sqrt{F}}{2}\right). \tag{29}$$

ϕ_n denotes the phase space Wigner function corresponding to the quantum mechanical state described by $|\psi_n\rangle$, i.e. the function to which the $|\psi_n\rangle\langle\psi_n|$ operator is mapped under the Weyl-Moyal correspondence ($\hbar = 1$). In our case

$$\phi_n(x) = 2(-1)^n e^{-|x|^2} L_n(2|x|^2), \tag{30}$$

L_n being the n-th Laguerre polynomial. These functions form the complete set of one-dimensional radially symmetric projectors solving the equation $\phi * \phi = \phi$[3]. The final answer reads

$$\begin{aligned} \big\langle :\bar{\phi}*\phi(x_{(1)})::\bar{\phi}*\phi(x_{(2)}):\big\rangle &= -\frac{1}{|1+F\theta|\pi^2}\left(\sum_{n=0}^{\infty}\frac{(-1)^n F\phi_n(\frac{x\sqrt{F}}{2})}{4((2n+1)F+m^2)}\right)^2, \\ x &\equiv x_{(1)} - x_{(2)}. \end{aligned} \tag{31}$$

In these calculations the two-point correlation function factors in a natural way just as in the commutative theory $\langle :\bar\phi\phi(x_{(1)})::\bar\phi\phi(x_{(2)}):\rangle = -|\langle\bar\phi(x_{(1)})\phi(x_{(2)})\rangle|^2$. So in the noncommutative case (31) is also a full square (not just a $*$-square) of a gauge invariant quantity. As for $F\to 0$ $\sum\limits_n \frac{(-1)^n F\phi_n(\frac{x\sqrt{F}}{2})}{4((2n+1)F+m^2)} \sim \sum\limits_n \frac{(-1)^n F\phi_n(\frac{x\sqrt{F}}{2})}{4m^2} = \frac{\delta^{(2)}(x)}{m^2}$ the correlator displays singular behaviour in this limit.

The higher correlators are calculated in the way similar to that of the commutative theory with the novel feature of multiplying Green's functions with the $*$-product like (23) and it is the $*$-product that does provide the noncommutative gauge invariance. For the evaluation of the n-point function one can rescale the integration variables so that

$$\frac{2+\alpha_2\theta}{1+\alpha_2\theta}k_i \to k_i, \tag{32}$$

then the Jacobian cancels the non-invariant factor coming from the rhs of (25) and every term of the series is explicitly gauge invariant (i.e. expressed in terms of F). Correlators with $n>2$ points do not reduce to projector solitons of[3] anymore, e.g. for $n=3$ there appear terms like

$$\begin{aligned}
&\int dk_1\, dk_2\, dk_3\, \mathrm{e}^{\frac{\mathrm{i}}{2}\left(x^1_{(1)}(k_3-k_1)+x^1_{(2)}(k_1-k_2)+x^1_{(3)}(k_2-k_3)\right)}\\
\times&\psi_{n_1}\left(x^2_{(1)}\sqrt{F}+\frac{(2+F\theta)k_1+F\theta k_3}{4\sqrt{F}}\right)\psi_{n_3}\left(x^2_{(1)}\sqrt{F}+\frac{(2+F\theta)k_3+F\theta k_1}{4\sqrt{F}}\right)\\
\times&\psi_{n_2}\left(x^2_{(2)}\sqrt{F}+\frac{(2+F\theta)k_2+F\theta k_1}{4\sqrt{F}}\right)\psi_{n_1}\left(x^2_{(2)}\sqrt{F}+\frac{(2+F\theta)k_1+F\theta k_2}{4\sqrt{F}}\right)\\
\times&\psi_{n_3}\left(x^2_{(3)}\sqrt{F}+\frac{(2+F\theta)k_3+F\theta k_2}{4\sqrt{F}}\right)\psi_{n_2}\left(x^2_{(3)}\sqrt{F}+\frac{(2+F\theta)k_2+F\theta k_3}{4\sqrt{F}}\right)
\end{aligned} \tag{33}$$

and there is no use trying to do a substitution like (26) unless $\theta=0$. It is easy to see that the latter expression does not vary when the identical shift in $x_{(i)}$'s is done so it depends only on the relative position of the points.

It is also useful to construct the generating functional. To do this we add to the action the current corresponding to the composite operator $\bar\phi*\phi$:

$$\begin{aligned}
S\to S+\int J(x)\,\bar\phi*\phi(x) &= S+\int A(x',x'')\bar\phi(x')\phi(x''),\\
A(x',x'') &= \int dx\, J(x)K(x,x',x'').
\end{aligned} \tag{34}$$

Then

$$Z[J] = N\det(\mathrm{i}G^{-1}+\mathrm{i}A) = \det(1+GA). \tag{35}$$

The generating functional for connected diagrams

$$\begin{aligned}
W[J] = \log Z[J] = \mathrm{tr}\log(1+GA) =\\
\mathrm{tr}(GA) - \frac{1}{2}\,\mathrm{tr}(GA)^2+\dots
\end{aligned} \tag{36}$$

Normal ordering $:\bar\phi*\phi:$ is nothing but removing the first term from the rhs manually. Obviously the former results are recovered because the variation $\frac{\delta A(x',x'')}{\delta J(x)}$ produces $K(x,x',x'')$ so after the integration over x' and x'' the $*$-product is reproduced.

4 Concluding remarks

All the previous results can be easily generalized to the more realistic case of 2+1-dimensional field theory in the constant magnetic field. The Green's functions are

$$G(x_{(1)},x_{(2)}) = -\frac{\mathrm{i}}{2}\sum_n \frac{\mathrm{e}^{-\mathrm{i}\sqrt{\lambda_n}|x^0_{(1)}-x^0_{(2)}|}}{\sqrt{\lambda_n}} f_n(\vec{x}_{(1)})\bar{f}_n(\vec{x}_{(2)}) \tag{37}$$

with the same f_n's. So the most interesting features survive. The 2-point gauge invariant functions can still be expressed in terms of the Wigner functions (noncommutative projector solitons). This statement seems to be valid for a large class of potentials.

The gauge invariance is recovered with the help of the $*$-product between the Green's functions that replaces the usual one and our result does depend on coordinates with *no* gauge-dependent rescaling resolving the seeming paradox (what naively does not look completely obvious).

The main goal of the paper is to verify the above physically nontrivial statements concerning noncommutative gauge invariance etc. explicitly.

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